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MOD p COHOMOLOGY OF SL(n, $\mathbb{Z}$)

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§1. INTRODUCTION

LET p be an odd prime, and let $\mathbb{F}_p$ be the field of p elements. This paper studies the mod p cohomology of $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. Our main result implies roughly that in almost every dimension, the mod p cohomology of $\mathrm{SL}(p-1, \mathbb{Z})$ grows faster than exponentially as a function of p.

The stable mod p cohomology—that is, $H^i(\mathrm{SL}(n, \mathbb{Z}); \mathbb{F}_p)$ for $n \gg i$—contains the algebra generated by Chern classes and the Euler class [9], together with the reductions mod p of the stable classes studied by Borel [5]. Little more is known in general, though progress has been made for certain values of i [2] [14]. At the other end of the spectrum, for i greater than the virtual cohomological dimension of Γ ($=\mathrm{vcd}\,\Gamma$), $H^i(\Gamma; M)$ is isomorphic to the Farrell cohomology $\hat{H}^i(\Gamma; M)$ for any coefficient module M. In principle, the Farrell cohomology is easier to compute than the ordinary cohomology, as for example in [4].

The present paper is primarily devoted to the cohomology in the range of dimensions between these two extremes. This should be important by analogy with what happens for automorphic forms associated to Γ and its congruence subgroups, where the complex cohomology becomes especially interesting in dimensions clustered around $\frac{1}{2}\mathrm{vcd}\,\Gamma$. Another reason for studying the mod p cohomology is its close relationship with the topological space B of lattices in $\mathbb{R}^n$ which possess an automorphism of order p. B is interesting in its own right, since one can study its Voronoi reduction theory (generalizing [15]) and related sphere-packing problems.

For the rest of this introduction, all cohomology groups have coefficients in the trivial module $\mathbb{F}_p$. Let $N = \frac{1}{2}n(n+1) - 1$ be the dimension of the symmetric space X associated to $\mathrm{SL}(n, \mathbb{R})$. Our main theorem is that for $n = p - 1$, and $\frac{1}{2}(p-1) \leqslant i \leqslant N - 1 - \frac{1}{2}(p-1)$, the canonical map

$$H^i(\Gamma) \oplus H^{N-1-i}(\Gamma) \to \hat{H}^i(\Gamma) \oplus \hat{H}^{N-1-i}(\Gamma)$$

is a supersemijection (the dimension of the image is $\geqslant$ one half the dimension of the target space). In other words,

$$\dim(H^i(\Gamma) \oplus H^{N-1-i}(\Gamma)) \geqslant \tfrac{1}{2}\dim(\hat{H}^i(\Gamma) \oplus \hat{H}^{N-1-i}(\Gamma))$$

in that range of i. Since $N \sim \frac{1}{2}p^2$, this range of i's is rather wide. The right hand side of the formula is very large (see (*) below in this introduction).

We now summarize the contents of the individual sections. The basic idea of this paper is to compare the cohomology of Γ, the Γ-equivariant cohomology of B, and the Farrell

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cohomology of Γ. The comparison of the latter two occupies sections 2 and 4, and works for general n. The result is stated as Theorem 4.1; it gives an isomorphism $H_\Gamma^i(B) \cong \hat{H}^i(\Gamma)$ for $i > d$, where $d = \dim B$. (For example, if $n = p - 1$, then $d = \frac{1}{2}(p - 3)$ is much smaller than n.) Section 3 is technical; we construct a neighborhood T of B such that B is a Γ-equivariant deformation retract of T and $X - T$ is a Γ-equivariant deformation retract of $X - B$. The methods here are general applications of facts about equivariant triangulations. In section 5 we derive our main theorem (5.5), using basic algebraic topology of manifolds (or more precisely, of V-manifolds); the extremely small dimension of B relative to X plays a crucial role.

The results of section 5 are stated only for $n = p - 1$. Our methods can be generalized to larger n, but there is a difficulty in that if $n > p - 1$, then B is not closed in the Borel–Serre bordification of X. The difficulty can be overcome by carefully considering how the closure of B meets the Borel-Serre boundary. We hope to do this in a later paper.

For $n \leqslant 2p - 3$, the mod p Farrell cohomology of $\mathrm{GL}(n, \mathbb{Z})$ was computed in [4]. In (6.1)–(6.2) we modify this calculation to find the mod p Farrell cohomology of $\mathrm{SL}(p - 1, \mathbb{Z})$ (we need SL to make our quotient spaces $\Gamma\backslash X$ orientable). As a result, we can state the following corollary (6.4), giving an explicit lower bound on the mod p Betti numbers for $\mathrm{SL}(p - 1, \mathbb{Z})$: for $\frac{1}{2}(p - 1) \leqslant i \leqslant N - 1 - \frac{1}{2}(p - 1)$, $\dim(H^i(\Gamma) \oplus H^{N-1-i}(\Gamma))$ is $\geqslant$ a quantity asymptotic to

$$(*) \qquad \frac{1}{(p-1)^2} \cdot 2^{(p+1)/2} \cdot p^{(p-1)/4}$$

for large p.

It is easy to see that the contribution of the Chern, Euler and Borel classes to the left-hand side of the preceding formula is minuscule in comparison to the right-hand side.

We expect our methods would work in studying the mod p cohomology of other arithmetic groups.

§2. FIXED POINT SETS OF $\mathbb{Z}/p\mathbb{Z}$ ACTING ON SYMMETRIC SPACES

2.1. Fix an odd prime p and an integer $n \geqslant p - 1$. Set $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. Γ acts on the right on V, the linear space of n-dimensional row vectors, so it acts on the left on the symmetric space $X = \{$positive-definite symmetric real $n \times n$ matrices$\}/$homotheties. We can view X as an open subset of the projective space $\mathbb{P}^N$ on $\mathrm{Sym}^2(V(\mathbb{R})^*)$, where $*$ denotes linear dual and $N = \frac{1}{2}n(n + 1) - 1$.

2.2. LEMMA. *Let C be a finite subgroup of Γ and let $F(C)$ be the fixed points of C in X. Then $F(C)$ is the intersection of a linear subspace $H \subseteq \mathbb{P}^N$ with X, with* $\dim H = \dim F(C) = \dim\{$*fixed points of C in* $\mathrm{Sym}^2(V(\mathbb{R})^*)\} - 1$.

Proof. The fixed points of C in $\mathrm{Sym}^2(V(\mathbb{R})^*)$ form a real sub-vector space which projects to H in $\mathbb{P}^N$. There exists some fixed point of C in X because we can average any C-orbit of positive-definite matrices. Then $\dim H = \dim F(C)$ because X is open in $\mathbb{P}^N$. ∎

2.3. Define $d(C) = \dim F(C)$.

LEMMA. *Let $C \subset \Gamma$ be a cyclic group of order p, and let $m = \dim_{\mathbb{C}} V(\mathbb{C})^C$. Then there is a non-negative integer k such that $n = k(p - 1) + m$ and*

$$d(C) = \tfrac{1}{2}k^2(p - 1) + \tfrac{1}{2}m(m + 1) - 1.$$

Proof. We compute $d(C)$ by looking for the 1-eigenspace on $\mathrm{Sym}^2(V(\mathbb{C})^*)$ of a generator $\gamma \in C$. Since γ is defined over $\mathbb{Z}$, its eigenvalues on $V(\mathbb{C})^*$ must be stable under the automorphisms of $\mathbb{C}$. Hence, if $\zeta = \exp(2\pi i/p)$, the list of eigenvalues consists of k copies of $\{\zeta, \zeta^2, \ldots, \zeta^{p-1}\}$, and m 1's. Then in $\mathrm{Sym}^2(V(\mathbb{C})^*)$ we get an eigenvalue 1 in two possible ways:

(1) pair a 1 with a 1, in $\frac{1}{2}m(m+1)$ possible ways;
(2) pair a ζ^j with a ζ^{-j}, there being $k(p-1)$ ways to choose the ζ^j, and k ways to choose a corresponding ζ^{-j}, times $\frac{1}{2}$. ($j \neq -j$ since $2 \nmid p$.) ∎

COROLLARY. *If $n = p - 1$, $d(C) = \dfrac{p-3}{2}$ for all subgroups $C \subset \Gamma$ of order p.*

Proof. Since C contains an element $\neq 1$, $m < \dim_{\mathbb{C}} V(\mathbb{C}) = p - 1$. This forces the set of eigenvalues to be exactly $\{\zeta, \ldots, \zeta^{p-1}\}$. So $k = 1$, $m = 0$. ∎

2.4. Let κ denote a conjugacy class of finite subgroups in Γ. Define $B(\kappa)$ to be the union of the $F(C)$, $C \in \kappa$. Clearly Γ acts on $B(\kappa)$.

LEMMA. *$B(\kappa)$ is a locally finite union of the flats $F(C)$ ($\forall C \in \kappa$) and Γ acts properly discontinuously on it.*

Proof. The second statement holds because Γ acts properly discontinuously on X. To prove the first statement, choose $x \in B(\kappa)$ and construct a neighborhood U of x in X such that $\mathscr{S} = \{\gamma \in \Gamma \mid U \cap \gamma \cdot U \neq \varnothing\}$ is finite. If a flat $F(C)$ meets U, then C fixes the points $F(C) \cap U$, hence $C \subseteq \mathscr{S}$. So for at most finitely many C can $F(C)$ meet U. ∎

Remark. It follows that $B(\kappa)$ is closed in X.

2.5. For the rest of the paper we will let κ be a conjugacy class of subgroups of Γ of order p, and we will often take $n \leqslant 2p - 3$. We now collect some information about these cases.

LEMMA. *Let $n \leqslant 2p - 3$. Then the only p-subgroups of Γ are cyclic of order p. There are only a finite number of conjugacy classes of them. If C and D are non-conjugate subgroups of order p in Γ, then $F(C) \cap F(D) = \varnothing$.*

Proof. All of this is standard except the last assertion. Suppose $x \in F(C) \cap F(D)$. Then C and D are contained in the stabilizer of x in Γ, which is a finite group. Thus C and D are p-Sylow subgroups in the same finite group, hence conjugate. ∎

For any n let K be the set of conjugacy classes of subgroups of order p in Γ and let B be the set of all points in X fixed by some element of order p in Γ.

COROLLARY. *Suppose $n \leqslant 2p - 3$. Then B is the disjoint union of $B(\kappa)$, $\kappa \in K$, and is a locally finite union of flat submanifolds in X.*

2.6. *Remark.* If $n = p - 1$, then B is contained in the well-rounded retract $W \subset X$ (see [3]—but note that in [3], what we are calling W was denoted by X). To see this, note that if a quadratic form θ is fixed by an element $\gamma \in \Gamma$ of order p, then γ acts on the set of its minimal non-zero integer vectors. Since γ has no eigenvalue ± 1 by the methods of (2.3), it fixes no minimal vector, meaning θ has at least p minimal vectors. These must span $\mathbb{R}^{p-1}$ (since $\mathbb{R}^{p-1}$ possesses no proper γ-invariant subspace defined over $\mathbb{Q}$); so θ is well-rounded.

$\Gamma\backslash B$ is therefore compact when $n = p - 1$, since $\Gamma\backslash B$ is a closed subset of the compact set $\Gamma\backslash W$. In particular, B is closed in the Borel–Serre bordification $\bar{X}$ of X and does not meet the boundary of $\bar{X}$. These ideas are discussed more in (5.2).

§3. TOPOLOGICAL PRELIMINARIES

Throughout this section, the notation X, B, Γ, $N = \dim X$, etc. is as in Section 2.

3.1. PROPOSITION. *There is a Γ-equivariant triangulation of X for which B is a full subcomplex.*

Proof. Let $\Gamma' \subset \Gamma$ be a torsion-free normal subgroup of finite index. The finite group Γ/Γ' acts smoothly on the smooth manifold $\Gamma'\backslash X$, so by [11] there is a Γ/Γ'-equivariant triangulation of $\Gamma'\backslash X$. Since $X \to \Gamma'\backslash X$ is a covering map, this lifts to a Γ-equivariant triangulation $\mathscr{T}$ of X.

B need not be a subcomplex of $\mathscr{T}$. But we claim B is a subcomplex of $\mathscr{T}'$, the first barycentric subdivision of $\mathscr{T}$. To see this, choose $x \in B$ and let $\sigma \in \mathscr{T}'$ be a simplex containing x in its interior. Let γ be an element of order p fixing x. Then γ must fix σ setwise because of the equivariance. But an element of Γ fixes an element of $\mathscr{T}'$ setwise if and only if it fixes it pointwise. So σ and all its faces are in B.

Replacing $\mathscr{T}'$ by the second barycentric subdivision $\mathscr{T}''$ guarantees B will be full. ∎

3.2. PROPOSITION. *There is a closed neighborhood T of B in X such that T strongly deformation-retracts onto B and $X - B$ strongly deformation-retracts onto $X - T^0$ (T^0 = interior of T). The retractions are Γ-equivariant. Both T and $X - T^0$ are N-dimensional manifolds with boundary, and the boundary of each is ∂T.* ∎

Proof. Take a triangulation $\mathscr{T}$ of X having B as a full subcomplex, as in (3.1). Take the second barycentric subdivision $\mathscr{T}''$ of $\mathscr{T}$, and define

$$T = \bigcup \{\bar{\sigma} \mid \sigma \in \mathscr{T}'', \bar{\sigma} \text{ meets } B\}$$

where $\bar{\sigma}$ is the union of the open simplex σ and all its faces. It is well known that T is a regular neighborhood of B in X: that is, T is a closed neighborhood of B which strongly deformation-retracts onto B, and T is an N-dimensional manifold with boundary [10, pp. 42–64, 89]. It follows that ∂T is a manifold [10, p. 27]. To see that $X - T^0$ is a manifold with boundary ∂T, use [10, p. 37].

To show that the retraction $T \to B$ is Γ-equivarient, we mildly generalize [10]. Take the set $\mathscr{N}$ of (open) simplices of $\mathscr{T}'$ whose closures meet B. For $\tau \in \mathscr{N}$ with $\dim \tau = l$, it is shown in [10, pp. 55–56] that $\bar{\tau} \cap T$ is a convex cell which retracts onto $\bar{\tau} \cap T \cap ((l-1)$-skeleton of $\mathscr{T}')$. Now partition $\mathscr{N}$ into Γ-orbits $\{\tau_{11}, \tau_{12}, \dots\}, \{\tau_{21}, \tau_{22}, \dots\}, \dots$ in order of decreasing dimension; that is, $\dim \tau_{ij} = l_i$ independently of j, and we have $l_i \geq l_{i+1}$ for all i. The equivariant retraction goes as follows: for $i = 1, 2, \dots$, simultaneously retract away $T \cap (\bar{\tau}_{i1} \cup \bar{\tau}_{i2} \cup \dots)$ onto the $(l_i - 1)$-skeleton in a Γ-equivariant way. There are two things to check to see that this is well-defined. First, if τ and τ' are two distinct simplices (of dimension l) in the same orbit, then the retractions of $\bar{\tau} \cap T$ and $\bar{\tau}' \cap T$ are compatible on $(\bar{\tau} \cap \bar{\tau}') \cap T$, for the simple reason that $T \cap ((l-1)$-skeleton of $\mathscr{T}')$ is being left fixed throughout the whole retraction. Second, if $\tau = \gamma \cdot \tau$ setwise for $\tau \in \mathcal{N}$ and $\gamma \in \Gamma$, then γ fixes τ pointwise since τ is in the first barycentric subdivision; in other words, if Γ_τ is the setwise stabilizer of τ, then Γ_τ acts trivially on τ, and the retraction of $\bar{\tau} \cap T$ is trivially Γ_τ-equivariant.

The retraction $X - B \to X - T$ is performed similarly. For each simplex σ of the first barycentric subdivision of B, let $v(\sigma)$ be the closure of $T \cap \{\tau \in \mathcal{T}' \mid \bar{\tau}$ meets the interior of $\sigma\}$. Fix an enumeration of the σ's in order of decreasing dimension, and take the corresponding enumeration $v_1, v_2, \ldots, v_k$ of the $v(\sigma)$'s. v_i retracts onto $v_i \cap (\partial T \cup v_{i+1} \cup \ldots \cup v_k)$. As in the preceding paragraph, we can make the retraction equivariant by retracting away whole Γ-orbits simultaneously. ∎

Remark. For both T and $X - T^0$ there are collarings of the boundary [10, p. 36]. They can be made Γ-equivariant by the methods of the preceding proof.

3.3. We recall the definition of a *V-manifold* [12]. Let Z be a Hausdorff space. A *local uniformizing system* for an open set $U \subseteq Z$ is a triple $(\tilde{U}, G, \varphi)$ where $\tilde{U}$ is a connected open subset of $\mathbb{R}^n$, G is a finite group of linear transformations $\tilde{U} \to \tilde{U}$ such that the set of all fixed points of G is of dimension $\leqslant n - 2$, and φ: $\tilde{U} \to U$ is a continuous map commuting with G and inducing a homeomorphism $G \backslash \tilde{U} \to U$. Z is a *V-manifold* of dimension n if it has a basis of open sets U such that each U has a local uniformizing system and the systems patch together (that is, when U and U' overlap, there is a homeomorphism between the appropriate pieces of $\tilde{U}$, $\tilde{U}'$ commuting with the groups and the φ's). A V-manifold is *orientable* if we can assign orientations to all the $\tilde{U}$ in such a way that they are preserved by all the patchings. The standard example: $\Gamma \backslash X$ is an orientable V-manifold of dimension N.

The notion of *V-manifold with boundary* of dimension n is defined similarly, by allowing $\tilde{U}$ to be either a connected open set in $\mathbb{R}^n$ or a connected open set in a closed half-space. The *boundary* of a V-manifold with boundary is well-defined (since a V-manifold itself has singularities only of codimension $\geqslant 2$); the boundary is a V-manifold of dimension $n - 1$. The boundary is orientable if the interior is.

A *p-manifold* Z of dimension n is a V-manifold in which $\forall x \in Z$, $H^*(Z, Z - x; \mathbb{F}_p)$ vanishes except in dimension n, where it has dimension 1. This is the case whenever $(\forall x \in Z)$ the isotropy group G_x associated to x has finite order prime to p. A *p-manifold* with boundary is a V-manifold with boundary for which both the interior and the boundary are p-manifolds.

3.4. PROPOSITION. *$\Gamma \backslash (X - T^0)$ is an orientable p-manifold with boundary.*

Proof. T^0, ∂T, and $X - T$ are all preserved by Γ. Since Γ acts properly discontinuously and with at most a finite stabilizer at any point, it is easy to show that $\Gamma \backslash (X - T^0)$ canonically inherits the structure of a V-manifold with boundary (compare [12, end of §2]). The quotient is orientable (and has orientable boundary), because SL$(n, \mathbb{R})$ preserves the orientation on X and therefore Γ does too.

By the definition of B, the order of the stabilizer in Γ of any point of $X - B$ (and hence of $X - T^0$) is not divisible by p. ∎

3.5. *Remark.* In general, B has several connected components, as shown in (2.5). T will fall into connected components in the corresponding way.

§4. COMPARISON OF COHOMOLOGY

In this section we prove that the p-primary part of the Γ-equivariant cohomology of B is naturally isomorphic to the p-primary part of the Farrell cohomology of Γ in degrees above $d = \dim B$. As pointed out to us by the referee, this is actually a general result, and we will state and prove it in general.

In this section only, let Γ be any group with $\mathrm{vcd}\,\Gamma < \infty$. Using the terminology of [8], let X be a finite dimensional, contractible, admissible, proper Γ-complex. Let $B \subset X$ be the set of points in X which are fixed by some element of order p in Γ. Let M be any $\mathbb{Z}\Gamma$-module used as coefficients for cohomology. We will suppress M in the notation. $\hat{H}$ denotes the Farrell cohomology theory [8, ch. X].

4.1. THEOREM. *We have a commutative triangle of natural maps*

$$\begin{tikzcd} & \hat{H}^i_\Gamma(X)_{(p)} \cong \hat{H}^i(\Gamma)_{(p)} \\ H^i(\Gamma)_{(p)} \cong H^i_\Gamma(X)_{(p)} \arrow[ur, "\varphi"] \arrow[dr, "\mathrm{res}"'] & \\ & H^i_\Gamma(B)_{(p)} \arrow[uu, "\psi"'] \end{tikzcd}$$

Moreover, ψ is an isomorphism for $i > d = \dim B$ and an epimorphism for $i = d$.

In the theorem, the subscript Γ denotes equivariant cohomology, the subscript (p) denotes p-primary part, φ is the natural map defined in [8, Chapter X], and ψ is described in the course of the proof.

Remarks. (1) If M is the trivial module $\mathbb{F}_p$, one can show ψ is actually an isomorphism in degree d and a monomorphism in degree $d - 1$.

(2) The hypotheses concerning Γ, X and B are satisfied by $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ and by X and B as defined in section 2. The necessary triangulations (which will be used during the proof) were constructed in (3.1).

4.2. *Proof of Theorem 4.1.* Imitating the top of p. 292 in [8], we have

$$\begin{aligned} \hat{H}^*(\Gamma)_{(p)} &\xleftarrow{\sim} \hat{H}^*_\Gamma(X)_{(p)} && \text{(because } X \text{ is contractible)} \\ &\xleftarrow{\sim} \hat{H}^*_\Gamma(B)_{(p)} && \text{(by Prop. VII.10.1 of [8] modified for } p\text{-primary parts).} \end{aligned}$$

We let ψ be the composition of the natural map $\alpha_{(p)}: H^*_\Gamma(B)_{(p)} \to \hat{H}^*_\Gamma(B)_{(p)}$ with these isomorphisms. Since the natural map from ordinary to Farrell cohomology commutes with restriction, the triangle commutes.

It remains to show that α is an isomorphism in degrees $> d$ and an epimorphism in degree d. Since we have a triangulation of X having B as a full subcomplex (3.1), we may compute $H^*_\Gamma(B)$ from the spectral sequence whose E_1-term is

$$E_1^{a,b} = \bigoplus_{\sigma \in \Sigma_b} H^a(\Gamma_\sigma) \Rightarrow H^{a+b}_\Gamma(B)$$

where Σ_b is a set of representatives modulo Γ of the b-dimensional cells of B and Γ_σ is the stabilizer of σ in Γ. There is a similar spectral sequence

$$\hat{E}_1^{a,b} = \bigoplus_{\sigma \in \Sigma_b} \hat{H}^a(\Gamma_\sigma) \Rightarrow \hat{H}^{a+b}_\Gamma(B)$$

and a natural map $E_1 \to \hat{E}_1$ compatible with the natural map $\alpha: H_\Gamma^*(B) \to \hat{H}_\Gamma^*(B)$ on the abutments. The assertion about α now follows immediately from a comparison of the two spectral sequences, which are isomorphic for $a \geqslant 1$ (since Γ_σ is always a finite group). ∎

§5. PROOF OF THE MAIN THEOREM

Note. Throughout the rest of the paper, all cohomology groups have coefficients in the trivial module $\mathbb{F}_p$. From (5.2) on, $n = p - 1$.

5.1. We have a Mayer–Vietoris sequence in Γ-equivariant cohomology:

$$\ldots \to \underbrace{H_\Gamma^i(X) \to H_\Gamma^i(X - T^0)}_{\eta_1} \oplus \underbrace{H_\Gamma^i(T) \to H_\Gamma^i(\partial T)}_{\eta} \to \ldots$$

The maps η_1, η are natural restriction maps.

LEMMA. *When $i < N - d - 1$, η_1 and η are isomorphisms.*

Proof. We prove this first for η_1. It suffices to show that

$$\pi_i(X \underset{\Gamma}{\times} E\Gamma, (X - B) \underset{\Gamma}{\times} E\Gamma) = 0 \quad \text{for} \quad i < N - d. \tag{5.1.1}$$

For then the Hurewicz theorem gives $H^i(X \times_\Gamma E\Gamma) \cong H^i((X - B) \times_\Gamma E\Gamma)$ for $i < N - d - 1$, meaning by definition $H_\Gamma^i(X) \cong H_\Gamma^i(X - B)$ for these i. $H_\Gamma^i(X - B) = H_\Gamma^i(X - T^0)$ because $X - B$ retracts to $X - T^0$ in a Γ-equivariant way (3.2).

We now prove (5.1.1). Choose $\xi \in \pi_i(X \times_\Gamma E\Gamma, (X - B) \times_\Gamma E\Gamma)$, and write it as a map $\xi: (I^i, \partial I^i) \to (X \times_\Gamma E\Gamma, (X - B) \times_\Gamma E\Gamma)$. Since $X \times E\Gamma \to X \times_\Gamma E\Gamma$ is a covering map and I^i is contractible, we can choose a lifting $\tilde{\xi}$ of ξ:

$$\begin{tikzcd} & X \times E\Gamma \arrow[d] \\ I^i \arrow[ur, "\tilde{\xi}"] \arrow[r, "\xi"'] & X \underset{\Gamma}{\times} E\Gamma \end{tikzcd}$$

By a dimension count, we can deform $\tilde{\xi}$ in the X-coordinates (leaving the $E\Gamma$-coordinates fixed) until its image lies entirely in $X - B$, and in such a way that $\tilde{\xi}(\partial I^i)$ never meets B during the deformation. Project the deformation downstairs. This proves (5.1.1).

The result for η follows from that for η_1 by excision of $X - T$. (The excision can be made rigorous using the collaring (3.2) of the boundary of $X - T^0$.) ∎

5.2. Up until now, most of our results have held for $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, $n \geqslant p - 1$. We now (and for the rest of the paper) specialize to the case $n = p - 1$.

Recall that Borel and Serre [6] have constructed a bordification $\bar{X}$ of X which is a manifold with corners. Using the notation of [6],

$$\bar{X} = X \amalg \coprod_P e(P),$$

where P runs over the non-trivial parabolic subgroups of $SL(n, \mathbb{Q})$. The $e(P)$ are called *boundary components*. Γ acts properly discontinuously on $\bar{X}$, with at most a finite stabilizer at any point; for any torsion-free $\Gamma' \subset \Gamma$ of finite index, $\Gamma'\backslash\bar{X}$ is a compact manifold with corners. It follows that $\Gamma\backslash\bar{X}$ is an orientable V-manifold with boundary. Γ preserves the decomposition of $\bar{X}$ into boundary components, and we call the $\Gamma\backslash e(P)$ the boundary components of $\Gamma\backslash\bar{X}$.

PROPOSITION. *For $n = p - 1$, there is a neighborhood of $\partial\bar{X}$ $(=\bar{X} - X)$ which does not meet T.*

Proof. As shown in (2.6), B is contained in the well-rounded retract and hence is bounded away from $\partial\bar{X}$. So T is also bounded away from $\partial\bar{X}$. ∎

Remark. For $n > p - 1$, this proposition is false, as we mentioned in the introduction. In a later paper we hope to consider the cases $p \leqslant n \leqslant 2p - 3$. Here T does meet $\partial\bar{X}$, and an understanding of $\partial T \cap \partial\bar{X}$ would yield information concerning the cohomology of Γ.

In (5.5) we will need the next

LEMMA. *$\Gamma\backslash(\partial\bar{X})$ is a p-manifold.*

Proof. If $g \in \Gamma$ stabilizes a point of the boundary component $e(P)$, then $g \in P$, and so g fixes a non-trivial flag of subspaces $\{0\} \subsetneq Q_1 \subsetneq \dots \subsetneq Q_k \subsetneq \mathbb{Q}^{p-1}$, $k \geqslant 1$. If g has order p, then its restriction to each subquotient $Q_1, Q_2/Q_1, \dots, \mathbb{Q}^{p-1}/Q_k$ must have order p or 1. The former is always impossible by an eigenvalue argument like that in (2.3), since each subquotient has dimension $< p - 1$. So g restricts to the identity on each subquotient, and a straightforward matrix computation then shows g cannot have order p. ∎

5.3. We will need a version of Lefschetz duality.

LEMMA. *Let M be a compact orientable N-dimensional p-manifold with boundary. Then we have a perfect pairing*

$$H^i(M, \partial M) \times H^{N-i}(M) \to \mathbb{F}_p$$

induced from the cup product.

Proof. It is easy to check that for any $x \in M$ there is a perfect pairing

$$H^i(U, \partial U; \mathbb{F}_p) \times H_c^{N-i}(U; \mathbb{F}_p) \to \mathbb{F}_p$$

induced from the cup product, where U is a contractible open neighborhood of x, $\partial U = U \cap \partial M$, and H_c^* is cohomology with compact support. A standard Mayer–Vietoris patching argument (see, for example, [7, pp. 45–46] or [12, §§6–9]) gives the general result. ∎

5.4. *Definition*: Let A, B be vector spaces, with B of finite even dimension. We say a linear map $f\colon A \to B$ is a *semijection* if $\operatorname{rank} f = \frac{1}{2}\dim B$. We say f is a *supersemijection* if $\operatorname{rank} f \geqslant \frac{1}{2}\dim B$.

PROPOSITION. *Let Y be any compact orientable p-manifold with boundary of dimension N. For each i, let* $\mathrm{res}_i\colon H^i(Y) \to H^i(\partial Y)$ *denote the natural restriction.*

(a) *For any i, $0 \leqslant i \leqslant N$,*

$$\mathrm{res}_i \oplus \mathrm{res}_{N-1-i}\colon H^i(Y) \oplus H^{N-1-i}(Y) \to H^i(\partial Y) \oplus H^{N-1-i}(\partial Y)$$

is a semijection.

(b) *Consider any decomposition $H^i(\partial Y) = V_1 \oplus \ldots \oplus V_q$, $H^{N-1-i}(\partial Y) = W_1 \oplus \ldots \oplus W_q$ where the pairing of* (5.3) *is perfect on $V_j \times W_j$ and is identically zero on $V_j \times W_{j'}$, for all $1 \leqslant j \neq j' \leqslant q$. Then the composition*

$$H^i(Y) \oplus H^{N-1-i}(Y) \xrightarrow{\mathrm{res}_i \oplus \mathrm{res}_{N-1-i}} H^i(\partial Y) \oplus H^{N-1-i}(\partial Y) \xrightarrow{\text{projection}} V_j \oplus W_j$$

is a supersemijection.

Proof. Consider the diagram

$$\begin{array}{ccccccc}
\ldots \to & H^i(Y) & \xrightarrow{\mathrm{res}_i} & H^i(\partial Y) & \longrightarrow & H^{i+1}(Y, \partial Y) & \to \ldots \\
& \times & & \times & & \times & \\
\ldots \leftarrow & H^{N-i}(Y, \partial Y) & \longleftarrow & H^{N-1-i}(\partial Y) & \xleftarrow{\mathrm{res}_{N-1-i}} & H^{N-1-i}(Y) & \leftarrow \ldots \\
& \downarrow & & \downarrow & & \downarrow & \\
& \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p &
\end{array}$$

where the outer vertical pairings are those of (5.3) and the middle one is Poincaré duality mod p on the p-manifold ∂Y. It is easy to check that every square "commutes": if some square is

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\times & & \times \\
C & \xleftarrow{g} & D \\
\downarrow & & \downarrow \\
\mathbb{F}_p & & \mathbb{F}_p
\end{array}$$

then $\forall a \in A$, $\forall d \in D$, $f(a) \cdot d = a \cdot g(d)$. A diagram chase shows that $(\mathrm{im}\ \mathrm{res}_i)^\perp = \mathrm{im}\ \mathrm{res}_{N-1-i}$, where im denotes image and $(\ )^\perp$ is with respect to the pairing. Part (a) follows immediately.

To prove (b), let r_j be res_i composed with the projection $H^i(\partial Y) \twoheadrightarrow V_j$, and let r'_j be res_{N-1-i} composed with the projection $H^{N-1-i}(\partial Y) \twoheadrightarrow W_j$ $(1 \leqslant j \leqslant q)$. We claim $(\mathrm{im}\, r_j)^{\perp_j} \subseteq \mathrm{im}\, r'_j$, where for $A \subseteq V_j$, $A^{\perp_j} = \{w \in W_j \mid \forall v \in A,\ v \cdot w = 0\}$. To see this, choose $w \in (\mathrm{im}\, r_j)^{\perp_j}$ (in particular, $w \in W_j$). Extend w by 0 to a vector $\tilde{w} = (\ldots, 0, w, 0, \ldots) \in H^{N-1-i}(\partial Y)$. For all $v \in H^i(Y)$, $r_j(v) \cdot w = 0$. Since $\tilde{w}$ is the extension by 0, $r_j(v) \cdot w = \mathrm{res}_i(v) \cdot \tilde{w}$ trivially. So $\tilde{w} \in (\mathrm{im}\ \mathrm{res}_i)^\perp$. By the proof of part (a), $\tilde{w} \in \mathrm{im}\ \mathrm{res}_{N-1-i}$. Again, since $\tilde{w}$ is w extended by 0, this means $w \in \mathrm{im}\, r'_j$. This proves the claim; part (b) follows immediately. ∎

5.5. We can now prove our main theorem.

THEOREM. *Let p be an odd prime. Let $\Gamma = \mathrm{SL}(p-1, \mathbb{Z})$. Let $N = \frac{1}{2}p(p-1) - 1$ as in* (2.1), *and let $d = \frac{1}{2}(p-3)$ as in* (2.3). *Then for any i with $d + 1 \leqslant i \leqslant N - d - 2$, the canonical map*

$$H^i(\Gamma; \mathbb{F}_p) \oplus H^{N-1-i}(\Gamma; \mathbb{F}_p) \to \hat{H}^i(\Gamma; \mathbb{F}_p) \oplus \hat{H}^{N-1-i}(\Gamma; \mathbb{F}_p)$$

is a supersemijection. Hence

$$\dim(H^i(\Gamma; \mathbb{F}_p) \oplus H^{N-1-i}(\Gamma; \mathbb{F}_p)) \geqslant \tfrac{1}{2}\dim(\hat{H}^i(\Gamma; \mathbb{F}_p) \oplus \hat{H}^{N-1-i}(\Gamma; \mathbb{F}_p))$$

for these i.

Proof. In (5.1) we considered the Mayer–Vietoris sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^i_\Gamma(X) \arrow[r, "\lambda^i"'] \arrow[rrr, bend left, "\tau^i"] & H^i_\Gamma(X - T^0) \arrow[rrr, bend right, "\theta^i"'] & \oplus & H^i_\Gamma(T) \arrow[r] \arrow[r, bend right, "\eta^i"'] & H^i_\Gamma(\partial T) \arrow[r] & \cdots
\end{tikzcd}$$

and proved that η^i is an isomorphism for the i considered in the theorem. In particular,

$$\tfrac{1}{2}\dim(H^i_\Gamma(T) \oplus H^{N-1-i}_\Gamma(T)) = \tfrac{1}{2}\dim(H^i_\Gamma(\partial T) \oplus H^{N-1-i}_\Gamma(\partial T))$$

and we call this quantity c. Because T retracts to B in a Γ-equivariant way, we deduce from (4.1) that τ^i can be identified with the canonical map from regular to Farrell cohomology and that

$$c = \tfrac{1}{2}\dim(\hat{H}^i(\Gamma) \oplus \hat{H}^{N-1-i}(\Gamma)).$$

LEMMA. *$\theta^i \oplus \theta^{N-1-i}$ is a supersemijection.*

Proof. First, note that $H^i_\Gamma(X - T^0) = H^i(\Gamma\backslash(X - T^0))$ and $H^i_\Gamma(\partial T) = H^i(\Gamma\backslash\partial T)$, because Γ acts on $X - T^0$ and on ∂T with finite stabilizers whose orders are prime to p.

Second, let $Y = \Gamma\backslash(\bar{X} - T^0)$. By Proposition 5.2, ∂Y is the disjoint union of $\Gamma\backslash(\partial\bar{X})$ and $\Gamma\backslash\partial T$. Lemma 5.2 says that $\Gamma\backslash(\partial\bar{X})$ is a p-manifold, and it follows easily that Y is a p-manifold with boundary. Then Proposition 5.4b implies that

$$H^i(Y) \oplus H^{N-1-i}(Y) \to H^i(\Gamma\backslash\partial T) \oplus H^{N-1-i}(\Gamma\backslash\partial T)$$

is a supersemijection.

Finally, each boundary component $e(P)$ of $\bar{X}$ has a collared neighborhood in $\bar{X}$; the collaring is Γ-equivariant, since it comes from the geodesic action and the latter commutes with Γ (see [6, §5]). We can remove the boundary components of $\Gamma\backslash\bar{X}$ one by one in order of increasing dimension, and because of the equivariant collaring we see that the $\mathbb{F}_p$-cohomology of the space does not change during this process. We conclude that

$$H^i(\Gamma\backslash(X - T^0)) \oplus H^{N-1-i}(\Gamma\backslash(X - T^0)) \to H^i(\Gamma\backslash\partial T) \oplus H^{N-1-i}(\Gamma\backslash\partial T) \qquad (5.5.1)$$

is a supersemijection. But this map is just $\theta^i \oplus \theta^{N-1-i}$. ∎

We continue the main proof. Let $W^i_\eta = (\eta^i)^{-1}(\operatorname{im}\theta^i)$ (recall that η^i is an isomorphism). Let W^i_θ be a subspace of $H^i_\Gamma(X - T^0)$ which θ^i maps isomorphically onto $\operatorname{im}\theta^i$. Let

$$W^i = \{a \oplus b \in H^i_\Gamma(X - T^0) \oplus H^i_\Gamma(T) \mid a \in W^i_\theta,\ b \in W^i_\eta,\ \theta^i(a) + \eta^i(b) = 0\}.$$

Clearly $W^i \cong W^i_\theta \cong W^i_\eta$ via maps induced from θ and η. In particular, the projection map $\mathrm{pr}^i_2\colon W^i \to W^i_\eta$ given by $a \oplus b \mapsto b$ is an isomorphism. Also,

$$\begin{aligned} \dim(W^i_\theta \oplus W^{N-1-i}_\theta) &= \operatorname{rank}(\theta^i \oplus \theta^{N-1-i}) \\ &\geqslant c \qquad \text{by (5.5.1).} \end{aligned} \qquad (5.5.2)$$

Clearly $\ker(\theta^i \oplus \eta^i) \supseteq W^i$, so $\mathrm{im}(\lambda^i) \supseteq W^i$. Thus $\mathrm{im}(\mathrm{pr}_2^i \circ \lambda^i) \supseteq \mathrm{pr}_2^i(W^i) = W_\eta^i$. But $\mathrm{pr}_2^i \circ \lambda^i$ is just τ^i, identified with the canonical map from regular to Farrell cohomology as above. So

$$\begin{aligned} \mathrm{rank}(\tau^i \oplus \tau^{N-1-i}) &\geqslant \dim(W_\eta^i \oplus W_\eta^{N-1-i}) \\ &= \dim(W_\theta^i \oplus W_\theta^{N-1-i}) \\ &\geqslant c \qquad \text{by (5.5.2)} \end{aligned}$$

which is the desired result. ∎

§6. LOWER BOUNDS FOR p-BETTI NUMBERS OF $\mathrm{SL}(p-1, \mathbb{Z})$

Notation. Throughout this section, $f(p) \sim g(p)$ means $\lim\limits_{p \to \infty} \dfrac{f(p)}{g(p)} = 1$.

6.1. The Farrell cohomology of $\mathrm{GL}(p-1, \mathbb{Z})$ with coefficients in $\mathbb{F}_p$ was computed in [4]. The result may be stated as follows:

Let L be the cyclotomic extension of $\mathbb{Q}$ generated by a primitive p-th root of unity, and Cl the ideal class group of L. For any $x \in \mathrm{Cl}$, let $s(x)$ denote the order of the stabilizer of x in $\mathscr{G} = \mathrm{Gal}(L/\mathbb{Q})$. Let $W(x, b, c)$ denote an $\mathbb{F}_p$-vector space whose dimension is the number of subsets $I \subseteq \{2, 4, \ldots, p-3\}$ with c elements such that $s(x)$ divides $\left(\left\lfloor \dfrac{b+1}{2} \right\rfloor + \sum_{i \in I} i\right)$, with $\lfloor \ldots \rfloor$ denoting the greatest integer function. For any $t \in \mathbb{Z}$, choose $T \in \mathbb{Z}$ such that $T \equiv t \pmod{2p-2}$ and $T > \frac{1}{2}(p-1)(p-2) = \mathrm{vcd}(\mathrm{GL}(p-1, \mathbb{Z}))$. Then for all $t \in \mathbb{Z}$,

$$\hat{H}^t(\mathrm{GL}(p-1, \mathbb{Z}); \mathbb{F}_p) \cong \bigoplus_{x \in \mathrm{Cl}/\mathscr{G}} \bigoplus_{\substack{b+c=T \\ c \geqslant 0}} W(x, b, c).$$

6.2. The computation of (6.1) can be adapted as follows for the Farrell cohomology of $\Gamma = \mathrm{SL}(p-1, \mathbb{Z})$. We start with Theorem X.7.4 in [8], which asserts that $\hat{H}(\Gamma; \mathbb{F}_p)$ is the direct sum over $\kappa \in K$ of $\hat{H}(N(C_\kappa); \mathbb{F}_p)$, where C_κ is a representative of the conjugacy class κ of subgroups of Γ of order p, and $N(C_\kappa)$ is its normalizer in Γ.

For $\mathrm{GL}(p-1, \mathbb{Z})$, these conjugacy classes are in one to one correspondence with $\mathscr{G}$-orbits in Cl. If x represents one of these $\mathscr{G}$-orbits, and C_x is a representative subgroup of order p in the corresponding conjugacy class, $N_{\mathrm{GL}(p-1, \mathbb{Z})}(C_x) \cong \mathrm{Stab}_{\mathscr{G}}(x) \ltimes \mathcal{O}^\times$, where $\mathcal{O}$ is the ring of integers in L [4, Thm. 2]. The right hand side is viewed as a subgroup of $\mathrm{GL}(p-1, \mathbb{Z})$ through its action on $\mathcal{O}$. Since the norm of any element in $\mathcal{O}^\times$ from L to $\mathbb{Q}$ is positive, the elements of $\mathcal{O}^\times$ all lie in Γ. An element in $\mathscr{G}$ lies in Γ if and only if it acts as an even permutation on the first $p-1$ powers of a primitive p-th root of 1, hence if and only if it lies in the subgroup $\mathscr{G}_0$ of order 2 in $\mathscr{G}$.

If two subgroups C, D of Γ of order p are conjugate in $\mathrm{GL}(p-1, \mathbb{Z})$, they are already conjugate in Γ if and only if $N_{\mathrm{GL}(p-1, \mathbb{Z})}(C)$ contains an element of determinant -1. Thus if $x \in \mathrm{Cl}$ and $\kappa(x)$ is the corresponding conjugacy class for $\mathrm{GL}(p-1, \mathbb{Z})$, $\kappa(x)$ will break up into two Γ-conjugacy classes if and only if $\mathrm{Stab}_{\mathscr{G}}(x) \subset \mathscr{G}_0$. Thus the first direct sum in the answer will range over $\mathrm{Cl}/\mathscr{G}_0$.

The next step is to compute the Farrell cohomology of $N_\Gamma(C_x) \cong \mathrm{Stab}_{\mathscr{G}_0}(x) \ltimes \mathcal{O}^\times$. We simply obtain the $\mathrm{Stab}_{\mathscr{G}_0}(x)$-invariant elements of $\hat{H}(\mathcal{O}^\times; \mathbb{F}_p)$, as opposed to the $\mathrm{Stab}_{\mathscr{G}}(x)$-invariant elements as computed in Theorem 5 of [4]. In sum, we get the following.

THEOREM. *For any* $x \in \mathrm{Cl}$, *let* $s_0(x)$ *denote the order of the stabilizer of* x *in* $\mathscr{G}_0$. *Let* $W_0(x, b, c)$ *denote an* $\mathbb{F}_p$*-vector space whose dimension is the number of subsets* $I \subseteq \{2, 4, \dots, p-3\}$ *with* c *elements such that* $s_0(x)$ *divides* $\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \sum_{i \in I} i\right)$. *For any* $t \in \mathbb{Z}$, *choose* $T \in \mathbb{Z}$ *such that* $T \equiv t \pmod{2p-2}$, $T > \frac{1}{2}(p-1)(p-2)$. *Then for all* $t \in \mathbb{Z}$,

$$\hat{H}^t(\Gamma; \mathbb{F}_p) \cong \bigoplus_{x \in \mathrm{Cl}/\mathscr{G}_0} \bigoplus_{\substack{b+c=T \\ c \geq 0}} W_0(x, b, c).$$

6.3. According to Theorem 6.2, we want to find a lower bound for

$$\dim \hat{H}^t(\Gamma; \mathbb{F}_p) = \sum_{x \in \mathrm{Cl}/\mathscr{G}_0} \sum_{b+c=T} \# \left\{ I \,\middle|\, I \subseteq \{2, 4, \dots, p-3\}, |I| = c, s_0(x) \middle| \left(\sum_{i \in I} i + \left\lfloor \frac{b+1}{2} \right\rfloor \right) \right\} \quad (6.3.1)$$

where the notation is as in the theorem. Since $b = T - c$ and $T = t + (2p-2)k$ for some k, we have $\left\lfloor \frac{b+1}{2} \right\rfloor = \left\lfloor \frac{t-c+1}{2} \right\rfloor + (p-1)k$. But for all x, $s_0(x)$ divides $|\mathscr{G}_0| = \frac{1}{2}(p-1)$ and hence divides $p-1$. So (6.3.1) becomes

$$\begin{aligned} \dim \hat{H}^t(\Gamma; \mathbb{F}_p) &= \sum_{x \in \mathrm{Cl}/\mathscr{G}_0} \sum_{c=0}^{\frac{1}{2}(p-3)} \# \left\{ I \,\middle|\, I \subseteq \{2, 4, \dots, p-3\}, |I| = c, \right. \\ &\qquad \left. s_0(x) \middle| \left(\sum_{i \in I} i + \left\lfloor \frac{t-c+1}{2} \right\rfloor \right) \right\} \\ &= \sum_{x \in \mathrm{Cl}/\mathscr{G}_0} \# \left\{ I \,\middle|\, I \subseteq \{2, 4, \dots, p-3\}, s_0(x) \middle| \left(\sum_{i \in I} i + \left\lfloor \frac{t-|I|+1}{2} \right\rfloor \right) \right\}. \quad (6.3.2) \end{aligned}$$

Note that this formula is periodic in t with period $p-1$, since $s_0(x)$ is always a divisor of $\left\lfloor \frac{p-1}{2} \right\rfloor = \frac{p-1}{2}$.

We now find a simple lower bound on (6.3.2). First, $|\mathrm{Cl}| = h = h_+ \cdot h_-$, where h_+ is a positive integer and

$$h_- \sim p^{(p-1)/4}$$

by [16, p. 44]. So $|\mathrm{Cl}/\mathscr{G}_0|$ is $\geq$ a quantity asymptotic to $\frac{2}{p-1} \cdot p^{(p-1)/4}$. Second, since $s_0(x) \left| \frac{p-1}{2} \right.$ for any x, each summand in (6.3.2) is bounded below by

$$f(p, t) = \# \left\{ I \,\middle|\, I \subseteq \{2, 4, \dots, p-3\}, s \middle| \left(\sum_{i \in I} i + \left\lfloor \frac{t-|I|+1}{2} \right\rfloor \right) \right\},$$

where $s = \frac{1}{2}(p-1)$. We conclude that for any given $t \in \mathbb{Z}$, $\dim \hat{H}^t(\Gamma; \mathbb{F}_p)$ is bigger than a quantity asymptotic to

$$\frac{2}{p-1} \cdot p^{(p-1)/4} \cdot f(p, t). \quad (6.3.3)$$

6.4. We now use (6.3.3) to bound the p-Betti numbers of Γ.

PROPOSITION. $f(p, t) \sim \frac{1}{p-1} 2^{(p-1)/2}$ *uniformly in* t *as* $p \to \infty$.

COROLLARY. *With notation as in (5.5), and for any i, $d + 1 \leqslant i \leqslant N - d - 2$, we have that* $\dim(H^i(\Gamma; \mathbb{F}_p) \oplus H^{N-1-i}(\Gamma; \mathbb{F}_p))$ *is $\geqslant$ a quantity which is asymptotic to*

$$\frac{1}{(p-1)^2} \cdot 2^{(p+1)/2} \cdot p^{(p-1)/4}$$

as $p \to \infty$.

The proof of the proposition occupies (6.5)–(6.6).

6.5. For any integer $\sigma \geqslant 1$ and integer c, $0 \leqslant c \leqslant \sigma - 1$, let $N_{\sigma,c}(k)$ be the number of ways to choose a subset $I \subseteq \{1, 2, 3, \ldots, \sigma - 1\}$ with $|I| = c$ such that $\sum_{i \in I} i \equiv k \pmod{\sigma}$.

LEMMA. *$N_{\sigma,c}(k)$ is asymptotic to $\frac{1}{\sigma}\binom{\sigma-1}{c}$ as $\sigma \to \infty$, uniformly in k and c. More precisely, for any $\varepsilon > 0$ there is an S such that for any c with $2 \leqslant c \leqslant \sigma - 3$, for any k, and for any $\sigma \geqslant S$,*

$$\frac{\left| N_{\sigma,c}(k) - \frac{1}{\sigma}\binom{\sigma-1}{c} \right|}{\frac{1}{\sigma}\binom{\sigma-1}{c}} < \varepsilon.$$

Remarks. (1) This is false for $c = 0, 1, \sigma - 2$ or $\sigma - 1$ for trivial reasons.

(2) The whole problem is symmetric under $c \mapsto (\sigma - 1) - c$, since

$$N_{\sigma,c}(k) = N_{\sigma,(\sigma-1)-c}(k_1 - k)$$

where k_1 is the constant $\sum_{i=1}^{\sigma-1} i$.

Proof of Lemma. It will be convenient to assume $3 \leqslant c \leqslant \sigma - 4$ in some of the estimates below. The cases $c = 2, \sigma - 3$ of the lemma can be handled directly. Also, by Remark 2 we may clearly take $c \leqslant \frac{\sigma-1}{2}$. So we will assume $3 \leqslant c \leqslant \frac{\sigma-1}{2}$ from now on.

Let $e\colon \mathbb{Z}/\sigma\mathbb{Z} \to \mathbb{C}^\times$ be a primitive character. For any $a \in \mathbb{Z}/\sigma\mathbb{Z}$, let $u(a) = \sigma/\gcd(\sigma, a)$ be the number of elements in the ideal $(a) \subseteq \mathbb{Z}/\sigma\mathbb{Z}$. Consider the polynomial

$$\prod_{l=1}^{\sigma-1} (1 - e(la)T) = \frac{1}{1-T}(1 - T^{u(a)})^{\frac{\sigma}{u(a)}}. \tag{6.5.1}$$

From the left-hand side the coefficient of T^c in (6.5.1) is

$$(-1)^c \sum_{k \,(\mathrm{mod}\, \sigma)} N_{\sigma,c}(k) \cdot e(ka).$$

The right-hand side of (6.5.1) is (letting u be shorthand for $u(a)$)

$$\begin{aligned}
&(1 + T + T^2 + \cdots + T^{u-1})(1 - T^u)^{\frac{\sigma}{u}-1} \\
&= (1 + T + T^2 + \cdots + T^{u-1}) \cdot \left(\sum_{j=0}^{\sigma-u} \left\{ \begin{matrix} 0 & \text{if } u \nmid j \\ (-1)^{\frac{j}{u}} \cdot \begin{pmatrix} \frac{\sigma}{u} - 1 \\ \frac{j}{u} \end{pmatrix} & \text{if } u \mid j \end{matrix} \right\} \cdot T^j \right) \\
&= \sum_{c=0}^{\sigma-1} (-1)^{[\frac{c}{u}]} \begin{pmatrix} \frac{\sigma}{u} - 1 \\ \left[\frac{c}{u}\right] \end{pmatrix} \cdot T^c.
\end{aligned}$$

Thus for all $a \in \mathbb{Z}/\sigma\mathbb{Z}$ we have

$$\sum_{k \pmod{\sigma}} N_{\sigma,c}(k) \cdot e(ka) = (-1)^{\lfloor \frac{c}{u} \rfloor - c} \begin{pmatrix} \frac{\sigma}{u} - 1 \\ \left\lfloor \frac{c}{u} \right\rfloor \end{pmatrix}.$$

Taking the Fourier transform, we have

$$N_{\sigma,c}(k) = \frac{1}{\sigma} \cdot \sum_{a \in \mathbb{Z}/\sigma\mathbb{Z}} (-1)^{\lfloor \frac{c}{u} \rfloor - c} \begin{pmatrix} \frac{\sigma}{u} - 1 \\ \left\lfloor \frac{c}{u} \right\rfloor \end{pmatrix} \cdot e(-ak).$$

Recall that u is a function of a. Since $u(a) = 1$ if and only if $a \equiv 0 \pmod{\sigma}$, we conclude that

$$\left| N_{\sigma,c}(k) - \frac{1}{\sigma}\binom{\sigma - 1}{c} \right| \leqslant \frac{1}{\sigma} \sum_{\substack{a \in \mathbb{Z}/\sigma\mathbb{Z} \\ a \neq 0}} \begin{pmatrix} \frac{\sigma}{u(a)} - 1 \\ \left\lfloor \frac{c}{u(a)} \right\rfloor \end{pmatrix}. \tag{6.5.2}$$

It is well known that we can extend the definition of binomial coefficient to real values by the formula $\binom{\alpha}{\beta} = \frac{\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta + 1)}$ for $\alpha \geqslant \beta \geqslant 0$, where $\Gamma(\cdot\cdot\cdot)$ is the gamma function. For fixed α this function strictly increases with β to its maximum at $\beta = \frac{1}{2}\alpha$, then strictly decreases symmetrically. For fixed β it strictly increases with α. Using these facts, and because $c \leqslant \frac{\sigma - 1}{2}$ and $u \geqslant 2$, we conclude that the right-hand side of (6.5.2) is $\leqslant \frac{\sigma - 1}{\sigma}\binom{\sigma/2}{c/2}$. So

$$\frac{\left| N_{\sigma,c}(k) - \frac{1}{\sigma}\binom{\sigma - 1}{c} \right|}{\frac{1}{\sigma}\binom{\sigma - 1}{c}} \leqslant (\sigma - 1)\frac{\binom{\sigma/2}{c/2}}{\binom{\sigma}{c}}.$$

A version of Stirling's formula with explicit estimates [1, p. 616] implies that

$$\frac{\left| N_{\sigma,c}(k) - \frac{1}{\sigma}\binom{\sigma - 1}{c} \right|}{\frac{1}{\sigma}\binom{\sigma - 1}{c}} < C_1 \cdot (\sigma - 1) \cdot \sqrt{\left(\frac{c}{\sigma}\right)^c \cdot \left(\frac{\sigma - c}{\sigma}\right)^{\sigma - c}} \tag{6.5.3}$$

where C_1 is a positive constant.

Let $f_1(\sigma, c)$ be the right-hand side of (6.5.3). Recall that we have assumed $3 \leqslant c \leqslant \frac{\sigma - 1}{2}$. Calculus shows that for any fixed σ, $f_1(\sigma, c)$ is strictly decreasing for $c \in \left(0, \frac{\sigma}{2}\right]$. So from (6.5.3), plugging in $c = 3$,

$$\begin{aligned} \frac{\left| N_{\sigma,c}(k) - \frac{1}{\sigma}\binom{\sigma - 1}{c} \right|}{\frac{1}{\sigma}\binom{\sigma - 1}{c}} &< C_2 \cdot \frac{\sigma - 1}{\sigma^{3/2}} \\ &\sim C_2 \cdot \sigma^{-1/2} \end{aligned}$$

(where C_2 is another positive constant). Since this can be made arbitrarily small for large enough σ, independently of the values of c and k, we are done. ∎

We will also need a related result. For any even integer $\sigma \geqslant 2$ and any integer c, $0 \leqslant c \leqslant \sigma - 1$, let $M_{\sigma,c}(k)$ be the number of ways to choose a subset $I \subseteq \{2, 4, \ldots, 2\sigma - 2\}$, $|I| = c$, such that $\sum_{i \in I} i \equiv k \pmod{\sigma}$. The notions of even and odd number extend to $\mathbb{Z}/\sigma\mathbb{Z}$; clearly $M_{\sigma,c}(k) = 0$ if $k \in \mathbb{Z}/\sigma\mathbb{Z}$ is odd.

LEMMA. *$M_{\sigma,c}(k)$ is asymptotic to $\dfrac{2}{\sigma}\dbinom{\sigma - 1}{c}$ for large σ whenever $k \in \mathbb{Z}/\sigma\mathbb{Z}$ is even, uniformly in c and k.*

More precisely, there is a constant c_0 such that for any $\varepsilon > 0$ there is an S such that for any c with $c_0 \leqslant c \leqslant \sigma - 1 - c_0$, for any even k, and for any $\sigma \geqslant S$,

$$\frac{\left|M_{\sigma,c}(k) - \dfrac{2}{\sigma}\dbinom{\sigma - 1}{c}\right|}{\dfrac{2}{\sigma}\dbinom{\sigma - 1}{c}} < \varepsilon.$$

Proof. Replace (6.5.1) with the formula

$$\prod_{l=1}^{\sigma-1} (1 - e(2la)T) = \frac{1}{1 - T}(1 - T^{u(a)})^{\frac{\sigma}{u(a)}-1}$$

where $u(a) = \sigma/\gcd(\sigma, 2a)$. Proceed as in the proof of the previous lemma. Note that here u takes the value 1 for *two* values of a, so that the main terms in the Fourier transform add up to $\dfrac{2}{\sigma}\dbinom{\sigma - 1}{c}$. ∎

6.6. *Proof of Proposition* 6.4. As above, let $s = \frac{1}{2}(p - 1)$.

Case of $p \equiv 3 \pmod 4$: $\{2, 4, 6, \ldots, p - 3\} = \{1, 2, 3, \ldots, s - 1\} \bmod s$ since s is odd, so we can apply the first lemma of 6.5. For each t, the sets $I \subseteq \{1, 2, \ldots, s - 1\}$ with c elements contribute

$$N_{s,c}\left(-\left\lfloor \frac{t - c + 1}{2} \right\rfloor\right)$$

to $f(p, t)$ (defined at the end of (6.3.)). This contribution is

$$\sim \frac{1}{s}\binom{s - 1}{c}.$$

For any $\varepsilon > 0$ there is an S such that for any $s \geqslant S$ and any t we have

$$\begin{aligned}
\frac{\left|f(p, t) - \dfrac{1}{s}2^{s-1}\right|}{\dfrac{1}{s}2^{s-1}} &\leqslant \sum_{c=0}^{s-1} \frac{\left|N_{s,c}\left(-\left\lfloor \dfrac{t - c + 1}{2} \right\rfloor\right) - \dfrac{1}{s}\dbinom{s - 1}{c}\right|}{\dfrac{1}{s}2^{s-1}} \\
&< \sum_{c=0}^{s-1} \frac{\varepsilon \cdot \dfrac{1}{s}\dbinom{s - 1}{c}}{\dfrac{1}{s}2^{s-1}} \qquad \text{by the lemma} \\
&= \varepsilon.
\end{aligned}$$

(We may ignore the cases $c = 0, 1, s-2, s-1$ since for these c, $\binom{s-1}{c}$ is negligible compared with 2^{s-1}.)

Since $\frac{1}{s}2^{s-1} = \frac{1}{p-1}2^{(p-1)/2}$, we are done with this case.

Case of $p \equiv 1 \pmod 4$: We apply the second lemma of (6.5), since s is even. For each t, the sets $I \subseteq \{2, 4, \ldots, p-3\}$ with c elements contribute

$$M_{s,c}\left(-\left\lfloor \frac{t-c+1}{2} \right\rfloor\right)$$

to $f(p, t)$. This contribution is

$$\left\{\begin{matrix} 0 \\ \\ \sim \frac{2}{s}\binom{s-1}{c} \end{matrix}\right\} \quad \text{if} \begin{Bmatrix} c \equiv t+2 \text{ or } t+3 \pmod 4 \\ c \equiv t+0 \text{ or } t+1 \pmod 4 \end{Bmatrix}.$$

As in the previous case, we use the error estimate of the lemma to conclude

$$f(p, t) \sim \frac{2}{s} \cdot \sum_{\substack{0 \leq c \leq s-1 \\ t-c+1 \equiv 0 \text{ or } 1 \pmod 4}} \binom{s-1}{c}.$$

So the proposition is implied by the following

LEMMA. *For any fixed $l \in \mathbb{Z}$ and $m \in \mathbb{N}$ (not necessarily even),*

$$\sum_{\substack{0 \leq c \leq m \\ c \equiv l \text{ or } l+1 \pmod 4}} \binom{m}{c} \sim 2^{m-1}$$

as $m \to \infty$.

Proof. This follows in a straightforward way from two facts.

(1) The basic theory of the binomial distribution [13, p. 62] says that for $a, b \in \mathbb{R}$,

$$\frac{1}{2^m}\sum \binom{m}{c} \sim \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx \qquad \text{as } m \to \infty$$

where the sum is over integers c, $0 \leq c \leq m$, such that $\frac{m}{2} + \frac{a\sqrt{m}}{2} < c \leq \frac{m}{2} + \frac{b\sqrt{m}}{2}$; furthermore, the convergence is uniform in a and b.

(2) Take $a_0 \in \mathbb{R}$, $q \in \mathbb{R}$, $q > 0$. Let $J(q) = \bigcup_{j \in \mathbb{Z}}\left[a_0 + \frac{2j}{q}, a_0 + \frac{2j+1}{q}\right]$. Then

$$\lim_{q \to \infty} \frac{1}{\sqrt{2\pi}}\int_{J(q)} e^{-x^2/2}\,dx = \frac{1}{2}.$$

This is proved by calculus, using the fact that $e^{-x^2/2}$ is uniformly continuous. ∎

6.7. *Remark.* As we mentioned in the introduction, $H^i(\Gamma; \mathbb{F}_p) \cong \hat{H}^i(\Gamma; \mathbb{F}_p)$ when $i > N - (n-1) = \operatorname{vcd}\Gamma$. So (6.3.3) shows that $H^i(\Gamma; \mathbb{F}_p)$ grows faster than exponentially as a function of p whenever i is above the vcd.

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REFERENCES

1. T. Apostol: *Calculus*, vol. II, Wiley, New York (1969).
2. D. Arlettaz: Chern-Klassen von ganzzahligen und rationalen Darstellungen diskreter Gruppen, *Math. Z.* **187** (1984), 49–60.
3. A. Ash: Cohomology of congruence subgroups of SL(n, $\mathbb{Z}$), *Math. Ann.* **249** (1980), 55–73.
4. A. Ash: Farrell cohomology of GL(n, $\mathbb{Z}$), *Israel. J. Math.* **67** (1989), 327–336.
5. A. Borel: Stable real cohomology of arithmetic groups, *Ann. Sci. Ec. Norm. Supér.* (4) 7 (1974), 235–272.
6. A. Borel and J.-P. Serre: Corners and arithmetic groups. *Comm. Math. Helvetici* **48** (1973), 436–491.
7. R. Bott and L. Tu: *Differential Forms in Algebraic Topology*, Springer-Verlag, New York (1982).
8. K. Brown: *Cohomology of Groups*, Springer-Verlag, New York, (1982).
9. B. Eckmann and G. Mislin: Galois action on algebraic matrix groups, Chern classes, and the Euler class, *Math. Ann.* **271** (1985), 349–358.
10. J. F. P. Hudson: *Piecewise Linear Topology*, mimeographed University of Chicago lecture notes, W. A. Benjamin, New York and Amsterdam, (1969).
11. S. Illman: Smooth equivariant triangulations of G-manifolds for G a finite group, *Math. Ann.* **233** (1978), 199–220.
12. I. Satake: On a generalization of the notion of manifold, *Proc. Natl. Acad. Sci. USA* **42** (1956), 359–363.
13. A. N. Shiryayev: *Probability*, Springer-Verlag, New York, (1984).
14. C. Soulé: K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale, *Inv. Math.* **55** (1979), 251–295.
15. G. Voronoi: Sur quelques propriétiés des formes quadratiques positives parfaites, *Crelle*, **133** (1908), 97–178.
16. L. Washington: *Introduction to Cyclotomic Fields*, Springer-Verlag, New York (1982).

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